Episode 15

Forced Vibrations Part 2
Base and Rotor Excitation

ENGN0040: Dynamics and Vibrations
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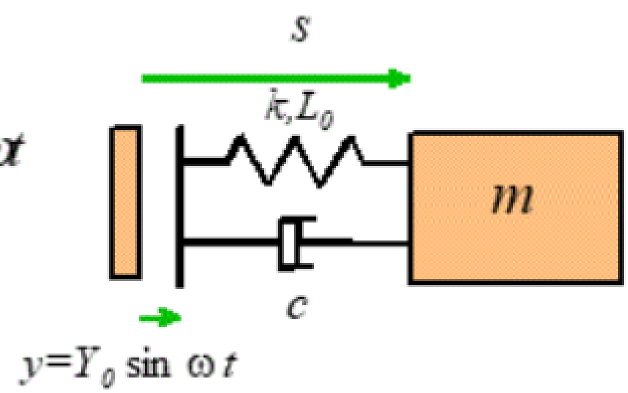
Topics for todays class

Forced Vibrations

1. Base excited harmonic oscillator
2. Examples
3. Rotor excited harmonic oscillator
4. Examples
5. Anti-resonant vibration isolation system

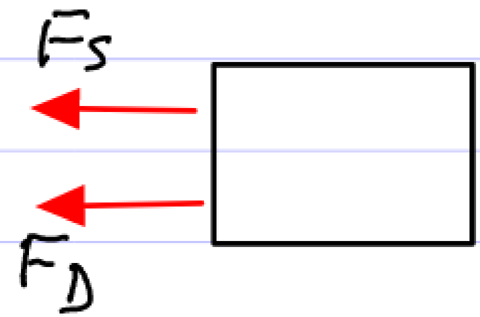
5.6.7 Base excited harmonic oscillator

Canonical base excited vibration problem: The base of the spring mass system moves harmonically $y(t) = Y_0 \sin \omega t$
Find steady state solution for *s(t)*.



Approach : (1) EOM ; (2) Solve (tables)

Equation of motion



$$F = ma \Rightarrow -F_s - F_d = m\frac{d^2 s}{dt^2}$$

$$F_s = k(s - y - L_0) \qquad F_d = c\frac{d}{dt}(s - y)$$

Hence

$$\boxed{\frac{m}{k}\frac{d^2 s}{dt^2} + \frac{c}{k}\frac{ds}{dt} + s = L_0 + y + \frac{c}{k}\frac{dy}{dt}}$$

List of standard ODEs for vibration problems

Our eq: $\frac{m}{k}\frac{d^2s}{dt^2} + \frac{c}{k}\frac{ds}{dt} + s = L_0 + y + \frac{c}{k}\frac{dy}{dt}$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = C + K\left(y + \frac{2\zeta}{\omega_n}\frac{dy}{dt}\right)$$

$K = 1$

Case I $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + x = C$

Case II $\quad \frac{1}{\alpha^2}\frac{d^2x}{dt^2} - x = -C$

Case III $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C$

Case IV $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C + KF(t)$ with $F(t) = F_0 \sin \omega t$

Case V $\quad \boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C + K\left(y + \frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right) \text{ with } y(t) = Y_0 \sin \omega t}$

Case VI $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\varsigma}{\omega_n}\frac{dx}{dt} + x = C - \frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$ with $y(t) = Y_0 \sin \omega t$

Case VII $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dx}{dt} + x = K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y\right)$ with $y(t) = Y_0 \sin \omega t$

$$\boxed{\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}} \qquad K = 1}$$

Solution to Case V (From pdf on website)

Equation $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$ Initial Conditions $x=x_0$ $\frac{dx}{dt}=v_0$ $t=0$

Full Solution $x(t)=C+x_h(t)+x_p(t)$

Focus on this

Steady state part (particular integral) $x(t)=X_0\sin(\omega t+\phi)$

$$X_0=\frac{KY_0\left\{1+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+(2\varsigma\omega/\omega_n)^2\right\}^{1/2}}\qquad \phi=\tan^{-1}\frac{-2\varsigma\omega^3/\omega_n^3}{1-(1-4\varsigma^2)\omega^2/\omega_n^2}$$

Transient part (complementary integral)

Overdamped $\varsigma>1$

$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{\frac{v_0^h+(\varsigma\omega_n+\omega_d)x_0^h}{2\omega_d}\exp(\omega_d t)-\frac{v_0^h+(\varsigma\omega_n-\omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$$

Critically Damped $\varsigma=1$

$$x_h(t)=\left\{x_0^h+\left[v_0^h+\omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

Underdamped $\varsigma<1$

$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{x_0^h\cos\omega_d t+\frac{v_0^h+\varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

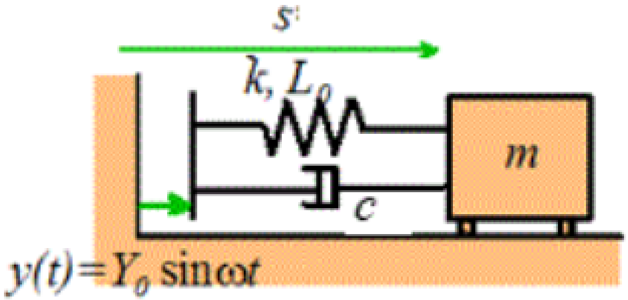
$$\omega_d=\omega_n\sqrt{|1-\varsigma^2|}\qquad x_0^h=x_0-C-x_p(0)=x_0-C-X_0\sin\phi\qquad v_0^h=v_0-\left.\frac{dx_p}{dt}\right|_{t=0}=v_0-X_0\omega\cos\phi$$

Steady state solution for base excited system

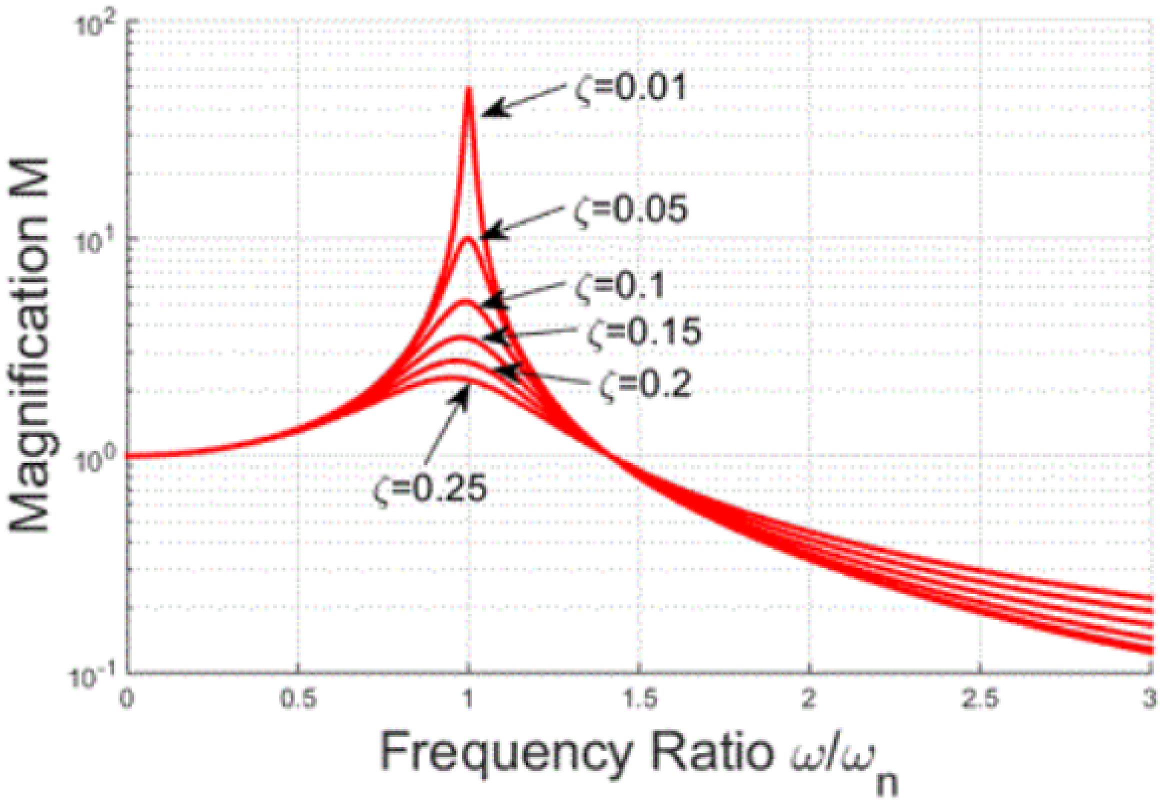
Steady state solution to $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$$

$$\omega_n=\sqrt{\frac{k}{m}}\quad \zeta=\frac{c}{2\sqrt{km}}\quad K=1$$

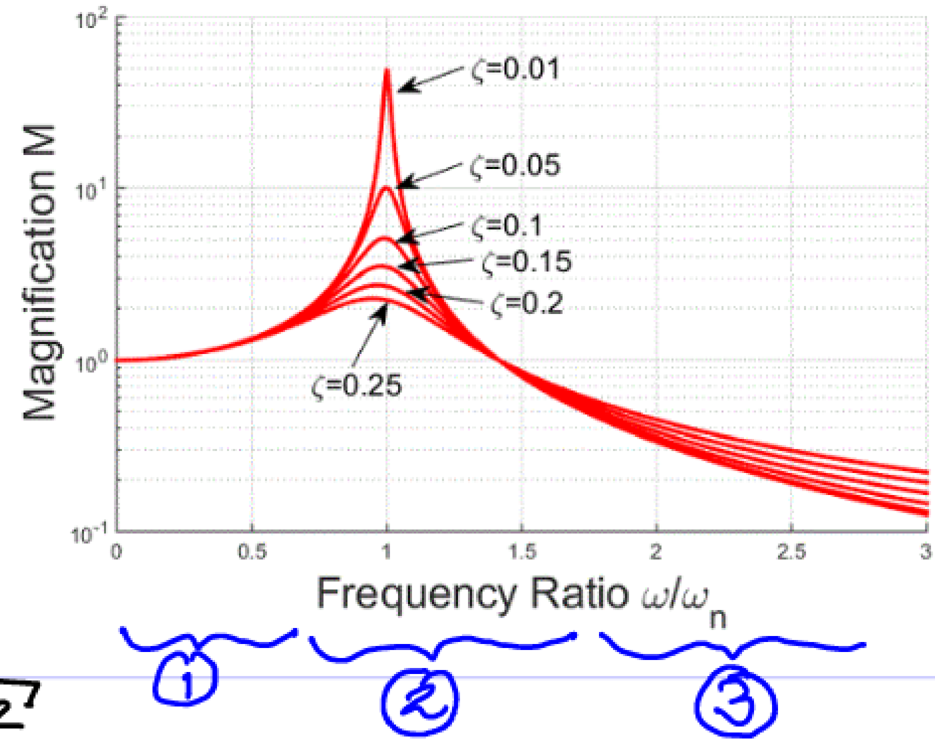
$$x(t)=X_0\sin(\omega t+\phi)$$



$$X_0=KY_0M(\omega,\omega_n,\zeta)\qquad M=\frac{\left\{1+(2\zeta\omega/\omega_n)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+(2\zeta\omega/\omega_n)^2\right\}^{1/2}}\qquad \phi=\tan^{-1}\frac{-2\zeta\omega^3/\omega_n^3}{1-(1-4\zeta^2)\omega^2/\omega_n^2}$$



Understanding Solution



$$x_p(t) = X_0 \sin(\omega t + \phi)$$

$$X_0 = K\, Y_0\, M(\omega/\omega_n, \zeta)$$

$$M = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n^2)^2+(2\zeta\omega/\omega_n)^2}}$$

(1) $\omega \ll \omega_n \Rightarrow M \approx 1 \Rightarrow X_0 = Y_0$

(2) $\omega \approx \omega_n \Rightarrow M \approx 1/2\zeta \Rightarrow X_0 = Y_0/2\zeta$
(Resonance)

(3) $\omega \gg \omega_n \quad M \approx 2\zeta\omega_n/\omega \Rightarrow X_0 = Y_0 \cdot 2\zeta\omega_n/\omega$

$\boxed{\text{Vibration isolation}}$

For isolation: (1) $\omega/\omega_n > \sqrt{2}$
(2) Small ζ is better

Steady-state solution to Case V equation

Solve: $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\zeta}{\omega_n}\frac{dy}{dt}\right)$ $\qquad y=Y_0\sin\omega t$

Let $y=Y_0\,\mathrm{Im}(e^{i\omega t})$ $\qquad$ Recall $\mathrm{Im}(z)=-i(z-\bar{z})/2$

Guess $x(t)=C+x_p(t)$ $\qquad x_p=MK\sin(\omega t+\phi)=MK\,\mathrm{Im}\left\{e^{i(\omega t+\phi)}\right\}$

Substitute into ODE $\qquad \left(1-\frac{\omega^2}{\omega_n^2}+i\frac{2\zeta\omega}{\omega_n}\right)Me^{i(\omega t+\phi)}=\left(1+i\frac{2\zeta\omega}{\omega_n}\right)e^{i\omega t}$

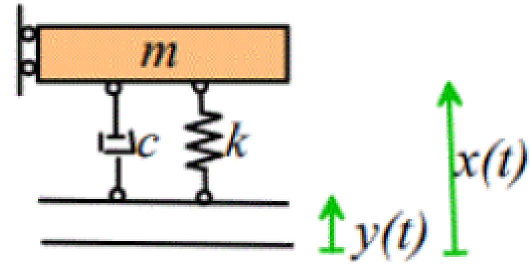
Hence $Me^{i\phi}=\dfrac{\left(1+i\frac{2\zeta\omega}{\omega_n}\right)}{\left(1-\frac{\omega^2}{\omega_n^2}+i\frac{2\zeta\omega}{\omega_n}\right)}$ $\qquad$ Euler: $a+ib=\rho e^{i\theta}$ $\qquad \rho=\sqrt{a^2+b^2}$ $\qquad \theta=\tan^{-1}(b/a)$

$$Me^{i\phi}=\frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}\,e^{\tan^{-1}(2\zeta\omega/\omega_n)}}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2+(2\zeta\omega/\omega_n)^2}\,e^{\tan^{-1}(2\zeta\omega/\omega_n/[1-\omega^2/\omega_n^2])}}$$

Hence $M=\dfrac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2+(2\zeta\omega/\omega_n)^2}}$ $\qquad \phi=\tan^{-1}(2\zeta\omega/\omega_n)-\tan^{-1}\left(\dfrac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right)$

5.6.8 Example: The vibration isolation system shown in the figure has m=20kg, k=19.8kN/m, c=1.259kN/m

The base vibrates harmonically with amplitude 1mm and frequency of 100Hz. What is the steady-state amplitude of vibration of the platform?



Use formulas for X_0

$$\omega_n = \sqrt{k/m} = 31.46 \text{ rad/s}$$

$$\zeta = c/(2\sqrt{km}) = 1$$

Magnification $$M = \frac{\sqrt{1+(2\zeta\omega/\omega_n)^2}}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} = 0.1$$

Vibration amplitude $X_0 = KMY_0$

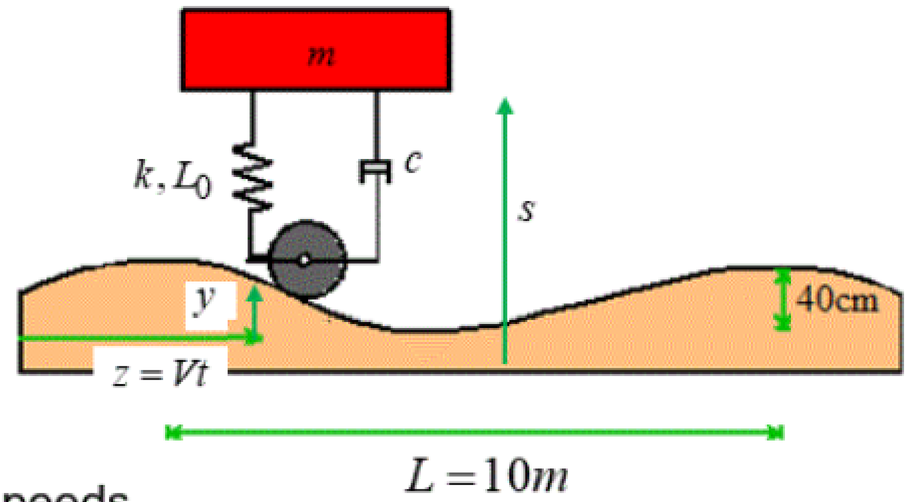
$K=1$ for base excited system

$$\Rightarrow \boxed{X_0 = 0.1 \text{ mm}}$$

5.6.9 Example: A car suspension has natural frequency $f_n = 2Hz$ and damping factor $\zeta = 0.2$
It drives over a road with a sinusoidal profile, with wavelength 10m and amplitude 20cm

(a) At what car speed does max vibration amplitude occur?
(b) What is the max vibration amplitude?

(c) Redesign the suspension. Constraints:
- Vibration amplitude must be less than 35cm at all speeds
- At 55mph, vibration amplitude must be less than 10cm
- Car weight 3000lb
- Select values for k and c



Approach: (1) Use road profile to find $y(t)$
(2) Max amplitude occurs if $\omega = \omega_n$

Road profile $y = Y_0 \sin \frac{2\pi z}{L}$ $\quad z = Vt$

$\Rightarrow y = Y_0 \sin \frac{2\pi V}{L} t$ $\quad \omega = \frac{2\pi V}{L}$

Max vibrations at $\omega = \omega_n$

$$\Rightarrow \frac{2\pi V}{L} = f_n \cdot 2\pi \Rightarrow \boxed{V = L f_n = 20\,m/s}$$

Vibration amplitude $X_0 = K M Y_0$

At $\omega = \omega_n$ $M \approx \frac{1}{2\zeta}$ $K = 1$

$$\Rightarrow \boxed{X_0 = 20/(2 \times 0.2) = 50\,cm}$$

To redisign the suspension

(1) Note X_0 is determined by M

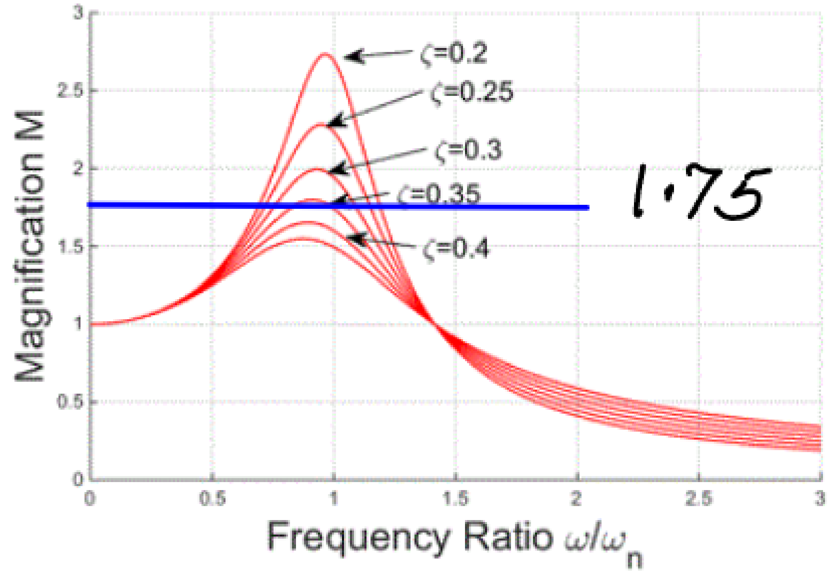
(2) Use constraints to find form needed for M

(3) Find ω_n, ζ to give desired M

(4) Use formulas for ω_n, ζ to find k, c

Constraints:

- Vibration amplitude must be less than 35cm at all speeds
- At 55mph, vibration amplitude must be less than 10cm



Formula: $X_0 = K\, M\, Y_0$
$(K = 1)$

Constraint (1): $X_0 < 35\text{cm}$ $\quad Y_0 = 20\text{cm}$

$$\Rightarrow M = \frac{X_0}{Y_0} < \frac{35}{20} = 1.75$$

From graph $\zeta > 0.38$ for $M < 1.75$

For best isolation we want ζ as small as possible

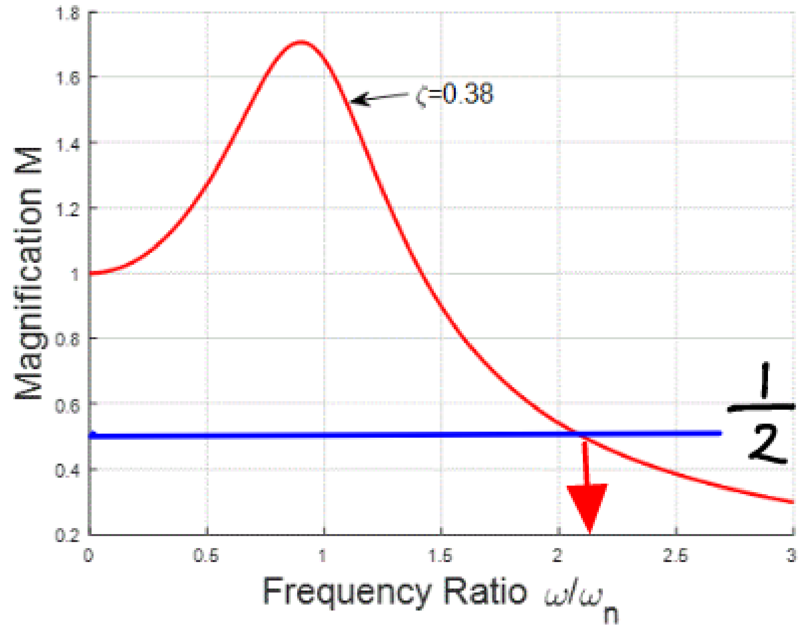
$\Rightarrow$ choose $\boxed{\zeta = 0.38}$

Constraint (2) $X_0 < 10$ cm for $V = 55$ mph (25 m/s)

Recall $\omega = \frac{2\pi V}{L} = \frac{2\pi \cdot 25}{10} = 5\pi$

$X_0 < 10 \Rightarrow \frac{X_0}{Y_0} < \frac{10}{20} \Rightarrow M < \frac{1}{2}$

We need $\frac{\omega}{\omega_n} > 2.1$ for $M < \frac{1}{2}$



Hence $\omega_n < \frac{\omega}{2.1} = \frac{5\pi}{2.1}$

Choose $\boxed{\omega_n = \frac{5\pi}{2.1}}$ (gives stiffest allowable suspension)

Finally recall $\omega_n = \sqrt{k/m}$

$$\Rightarrow \boxed{k = m\,\omega_n^2 = 168 \times 10^3 \text{ lb/ft}}$$

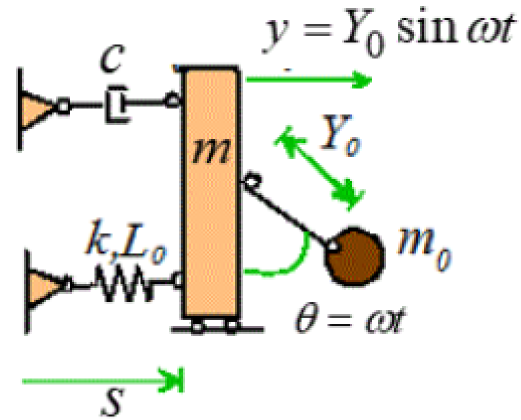
also $\zeta = c/(2\sqrt{km})$

$$\Rightarrow \boxed{c = 2\sqrt{km}\,\zeta = 17 \times 10^3 \text{ lb s/ft}}$$

5.6.10 Rotor excited harmonic oscillator

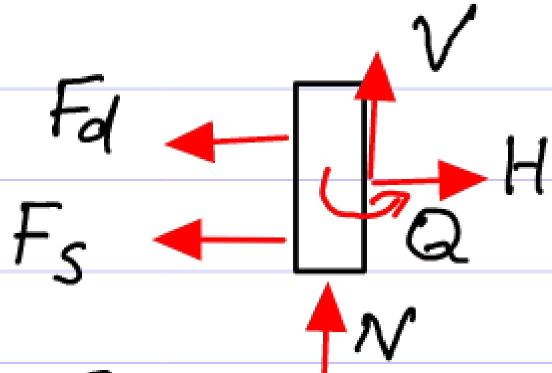
Canonical rotor excited vibration problem: A rotor with mass m_0 and length Y_0 rotates at steady angular velocity ω It is attached to mass *m* which is supported by a spring and damper.

Find steady state solution for *s(t)*.



$F_x = m a_x$ for mass m

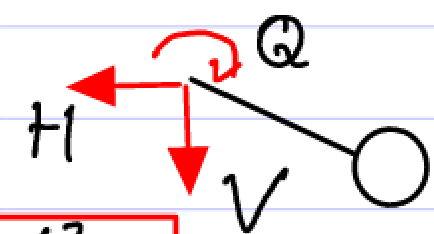
$$m \frac{d^2 s}{dt^2} = -F_S - F_d + H$$



$F_x = m a_x$ for mass m_0

$$m_0 \frac{d^2}{dt^2}(s+y) = -H$$

Add eqs: $(m+m_0)\frac{d^2 s}{dt^2} + m_0 \frac{d^2 y}{dt^2} = -F_S - F_d$



$F_S = k(s - L_0)$ $\qquad$ $F_d = c\, ds/dt$

$$\Rightarrow \boxed{\frac{m+m_0}{k}\frac{d^2 s}{dt^2} + \frac{c}{k}\frac{ds}{dt} + s = L_0 - \frac{m_0}{k}\frac{d^2 y}{dt^2}}$$

List of standard ODEs for vibration problems

$$\frac{m+m_0}{k}\frac{d^2s}{dt^2}+\frac{c}{k}\frac{ds}{dt}+s=\lambda_0-\frac{m_0}{k}\frac{d^2y}{dt^2}$$

$$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

Case I $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$

Case II $\quad \frac{1}{\alpha^2}\frac{d^2x}{dt^2}-x=-C$

Case III $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$

Case IV $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF(t)$ with $F(t)=F_0\sin\omega t$

Case V $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$ with $y(t)=Y_0\sin\omega t$

Case VI $\quad \boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2} \text{ with } y(t)=Y_0\sin\omega t}$

Case VII $\quad \frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right)$ with $y(t)=Y_0\sin\omega t$

$$\frac{K}{\omega_n^2}=\frac{m_0}{k}$$

$$\Rightarrow K=\frac{m_0}{k}\frac{k}{m+m_0}$$

$$\boxed{\omega_n=\sqrt{\frac{k}{m+m_0}} \qquad \zeta=\frac{c}{2\sqrt{k(m+m_0)}} \qquad K=\frac{m_0}{m+m_0}}$$

Solution to Case VI (From pdf on website)

Equation $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$$

Initial Conditions $x=x_0 \quad \frac{dx}{dt}=v_0 \quad t=0$

Full Solution $x(t)=C+x_h(t)+x_p(t)$

Focus on this

Steady state part (particular integral) $x_p(t)=X_0\sin(\omega t+\phi)$

$$X_0=KY_0M(\omega,\omega_n,\zeta) \qquad M=\frac{\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}} \qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

Transient part (complementary integral)

Overdamped $\varsigma>1$
$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{\frac{v_0^h+(\varsigma\omega_n+\omega_d)x_0^h}{2\omega_d}\exp(\omega_d t)-\frac{v_0^h+(\varsigma\omega_n-\omega_d)x_0^h}{2\omega_d}\exp(-\omega_d t)\right\}$$

Critically Damped $\varsigma=1$
$$x_h(t)=\left\{x_0^h+\left[v_0^h+\omega_n x_0^h\right]t\right\}\exp(-\omega_n t)$$

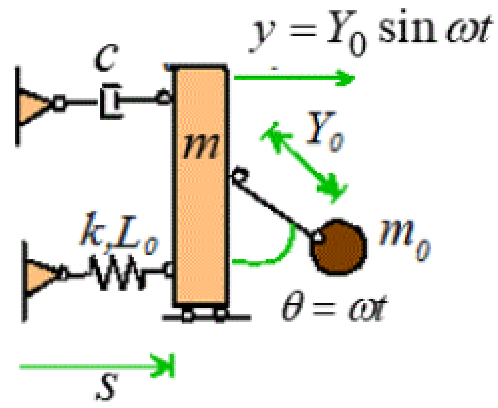
Underdamped $\varsigma<1$
$$x_h(t)=\exp(-\varsigma\omega_n t)\left\{x_0^h\cos\omega_d t+\frac{v_0^h+\varsigma\omega_n x_0^h}{\omega_d}\sin\omega_d t\right\}$$

$$\omega_d=\omega_n\sqrt{1-\varsigma^2} \qquad x_0^h=x_0-C-x_p(0)=x_0-C-X_0\sin\phi \qquad v_0^h=v_0-\left.\frac{dx_p}{dt}\right|_{t=0}=v_0-X_0\omega\cos\phi$$

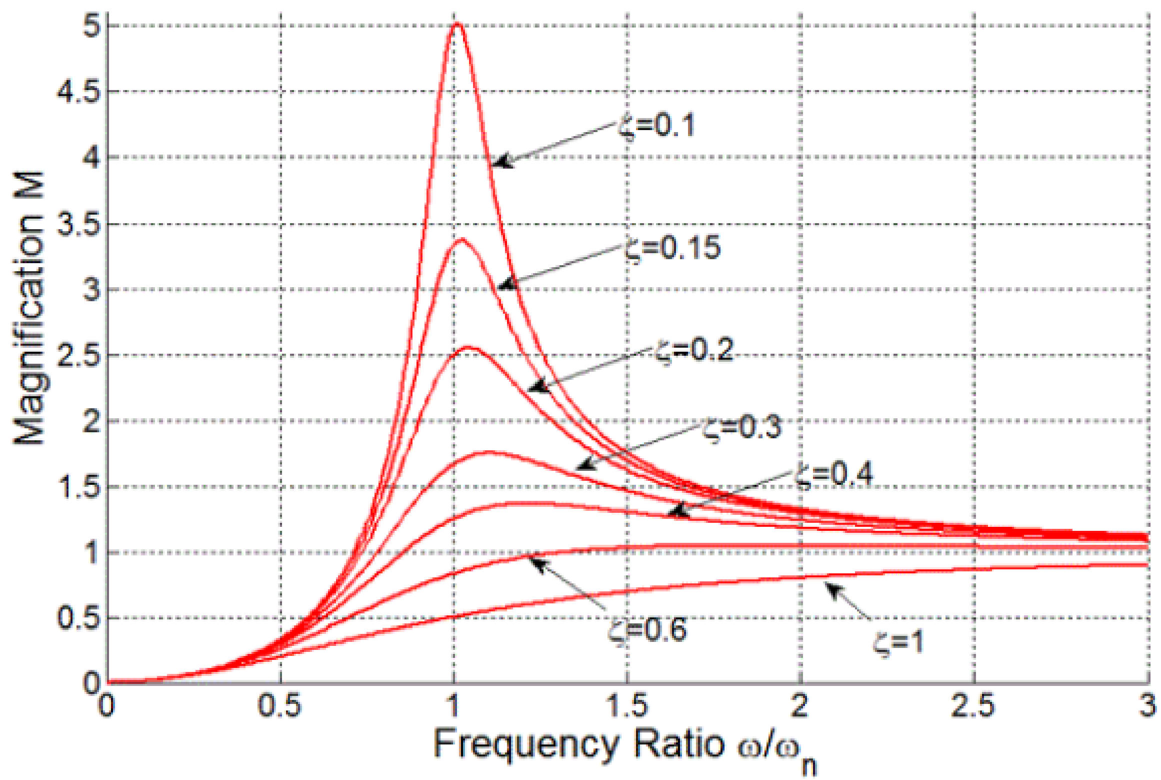
Steady state solution for rotor excited system

Steady state solution to $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$

$$\omega_n=\sqrt{\frac{k}{m+m_0}}\quad \zeta=\frac{c}{2\sqrt{k(m+m_0)}}\quad K=\frac{m_0}{m+m_0}\qquad x_{ss}(t)=X_0\sin(\omega t+\phi)$$



$$X_0=KY_0M(\omega,\omega_n,\zeta)\qquad M=\frac{\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\varsigma\omega/\omega_n\right)^2\right\}^{1/2}}\qquad \phi=\tan^{-1}\frac{-2\varsigma\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

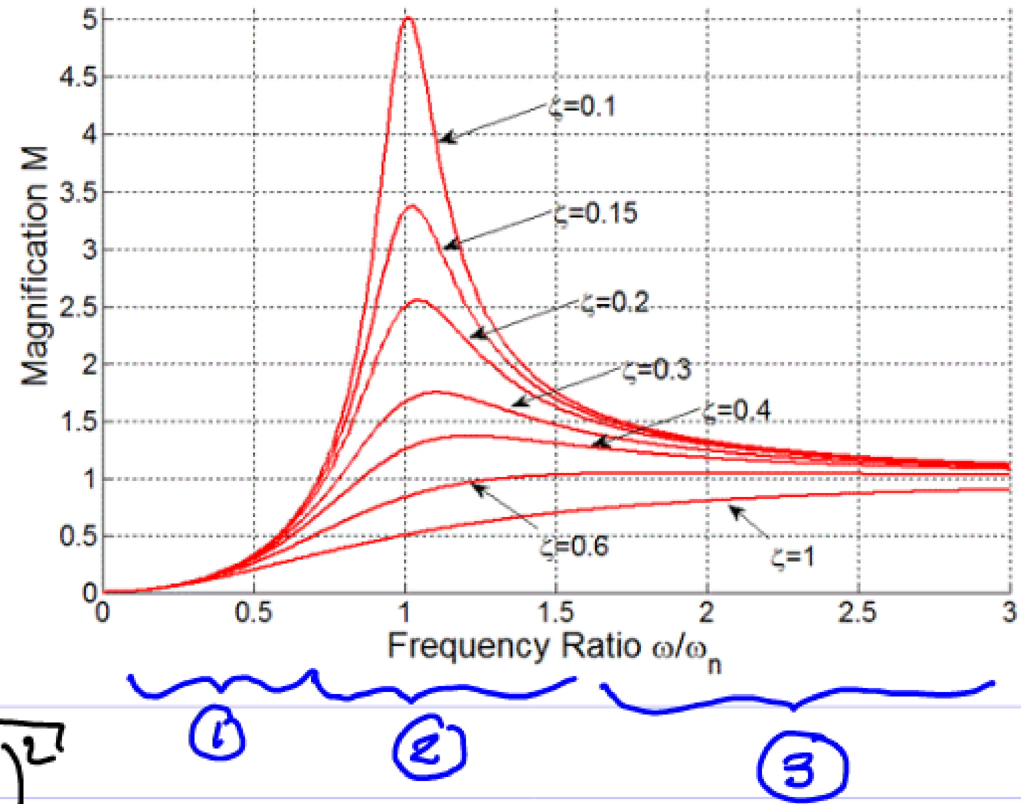


Understanding solution

$$x_p = X_o \sin(\omega t + \phi)$$

$$X_o = K\, Y_o\, M$$

$$K = m_0 / (m + m_0)$$



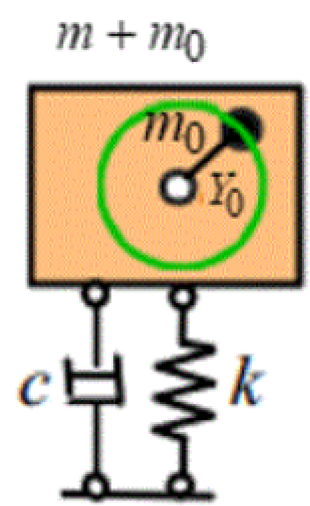
$$M = \frac{\omega^2/\omega_n^2}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

1 $\quad M \approx \omega^2/\omega_n^2 \Rightarrow X_o = (m_0 \omega^2/k)\, Y_o$

2 $\quad M \approx 1/2\zeta \Rightarrow X_o = [m_0/(m+m_0)]\,[Y_o/2\zeta]$
(resonance)

3 $\quad M \approx 1 \Rightarrow X_o = [m_0/(m+m_0)]\, Y_o$

5.6.11 Example: A motor with total mass $m+m_0$ =50kg has rotating internal mass of m_0 = 1kg that rotates on a shaft with eccentricity Y_0 =1mm at angular rate ω = 100rad/s. The engine is mounted on vibration isolation pads with effective stiffness *k*=500000N/m and a dashpot coefficient *c=250* Ns/m. The system is found to have a severe vibration problem. Which of the following changes will reduce the vibration amplitude?

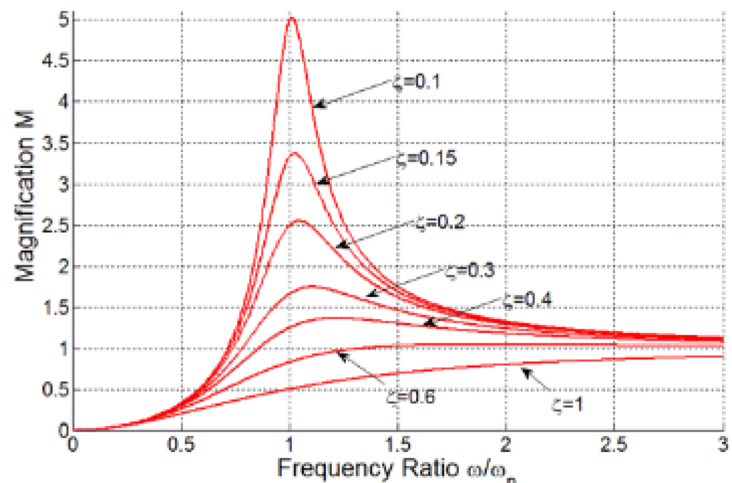


(a) Increase the pad stiffness *k*
(b) Decrease the pad stiffness *k*
(c) Increase the speed of the motor
(d) Decrease the dashpot coefficient *c*

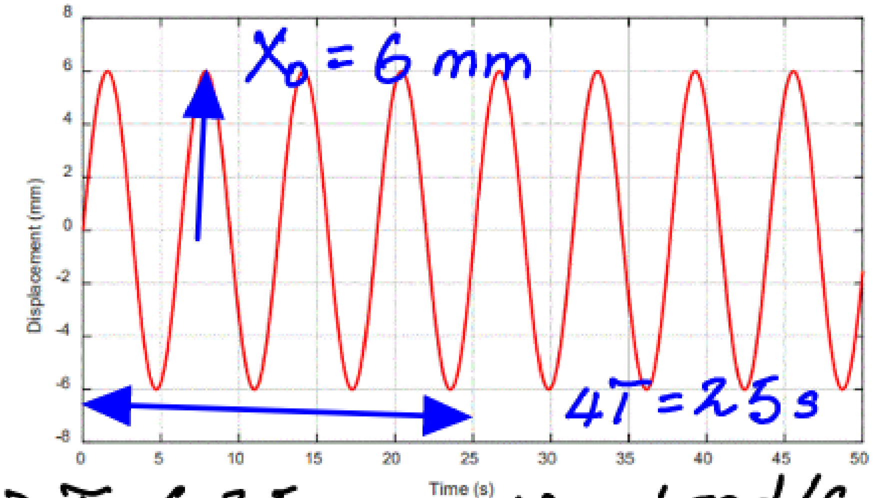
$\omega_n = \sqrt{k/(m+m_0)} = 100$ rad/s $\qquad \zeta = c/2\sqrt{k(m+m_0)} = 0.025$

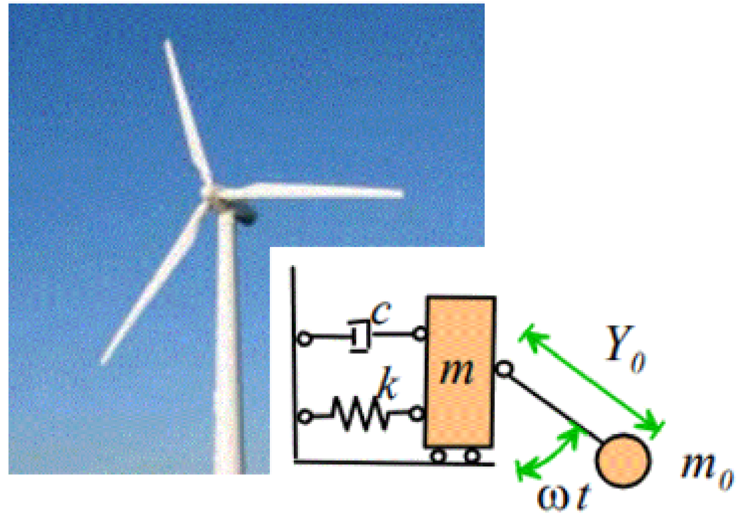
Hence $\omega/\omega_n = 1 \Rightarrow$ resonance!

(a) Increases $\omega_n \Rightarrow$ reduces amplitude
(b) Decreases $\omega_n \Rightarrow$ reduces amplitude
(c) Increases $\omega \Rightarrow$ " "
(d) Decreases $\zeta \Rightarrow$ increases amplitude



$\Rightarrow$ $\boxed{\text{(a), (b), (c) reduce amplitude}}$





$\Rightarrow T = 6.25 \Rightarrow \omega \sim 1$ rad/s

5.6.12 Example: An unbalanced wind turbine is idealized as a rotor excited spring-mass system. The mass m represents the tower, and m_0 represents the combined mass of the three rotor blades. The rotor is 'unbalanced' because its center of mass is a distance Y_0 away from the axle. The total mass of the turbine is 25000kg, the spring stiffness is 4100 kN/m and the dashpot coefficient is 128 kNs/m.

The figure shows the measured displacement of the system during operation. The blades have a radius of 40m. Assuming that the rotor can be balanced by adding mass to the tip of one blade, estimate the mass that must be added to balance the rotor.

Approach: (1) Balance rotor by moving COM to $\underline{r} = \underline{0}$

(2) Recall $\underline{r}_G = (\sum m_i \underline{r}_i) / (\sum m_i) \Rightarrow m_0 Y_0 - m^* R = 0$

(3) To find m^* we need to know $m_0 Y_0$

(4) Recall $X_0 = \underline{\varepsilon} Y_0 \underline{M} = [m_0 Y_0 / (m + m_0)] \underline{M}$

$\Rightarrow$ Find $m_0 Y_0$ using $\underline{X}_0$, $\underline{M}$, $(m + m_0)$ (all given)

5.6.12 Example: An unbalanced wind turbine is idealized as a rotor excited spring-mass system. The mass m represents the tower, and m_0 represents the combined mass of the three rotor blades. The rotor is 'unbalanced' because its center of mass is a distance Y_0 away from the axle. The total mass of the turbine is 25000kg, the spring stiffness is 4100 kN/m and the dashpot coefficient is 128 kNs/m.

$$\omega_n = \sqrt{k/(m+m_0)} = 12.8 \text{ rad/s}$$

$$\zeta = c/2\sqrt{k(m+m_0)} = 0.2$$

$$X_0 = R\, M\, Y_0 = \frac{m_0 Y_0}{(m+m_0)} \frac{\omega^2/\omega_n^2}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$$

$\omega = 1$ rad/s $\quad X_0 = 6\times10^{-3}$ m

$\Rightarrow \quad Y_0 m_0 = 25\times10^3$ kg m

To balance turbine add mass m^* at radius

$R = 40$ m $\qquad Y_0 m_0 + m^*(-R) = 0$

$$\Rightarrow \boxed{m^* = Y_0 m_0 / R = 625 \text{ kg}}$$

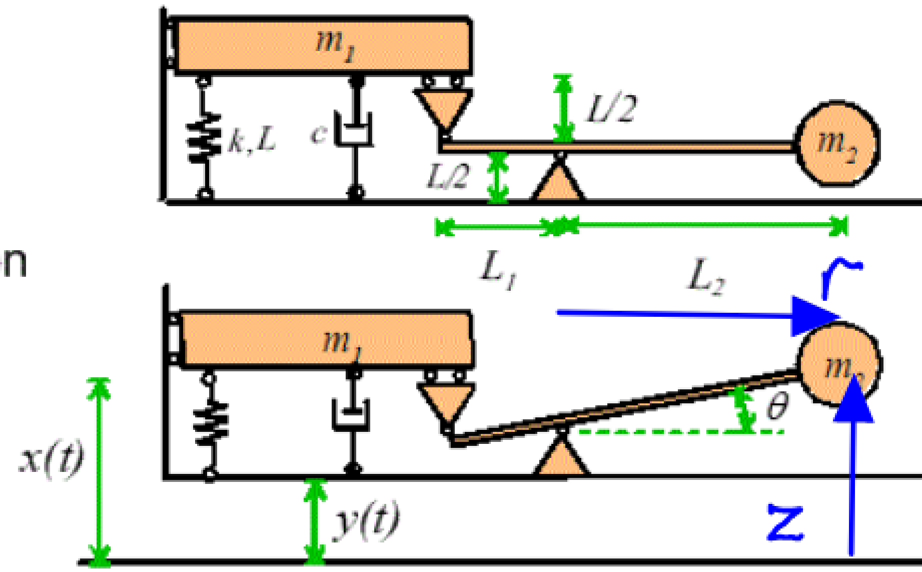
5.6.13 Example: The figure shows an 'anti-resonant' vibration isolation system.

(a) Find the equation of motion relating $x(t)$ to $y(t)$. Assume $\theta << 1$ and neglect gravity

(b) Plot the 'transmissibility' of the isolator as a function of frequency for

$$k = 20kN/m \quad m_1 = 1kg, \; m_2 = 10\,grams, \; c = 20Ns/m$$

$$L_2 / L_1 = 10$$



Equation of motion

Geometry: $z = y + L/2 + L_2 \sin\theta \approx y + L/2 + L_2\theta$

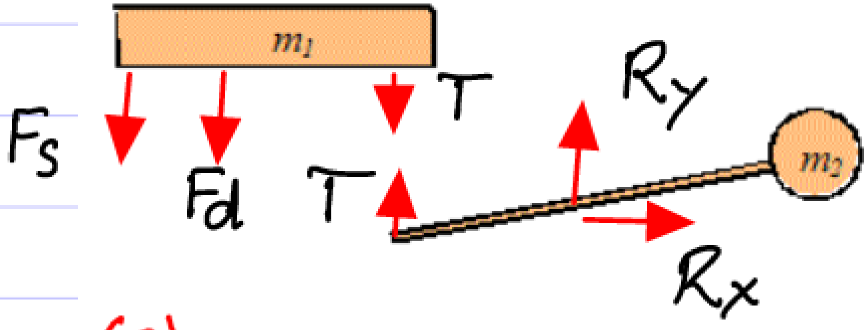
$$x = y + L - L_1 \sin\theta \approx y + L - L_1\theta$$

$$r = L_2 \cos\theta \approx L_2$$

$$\Rightarrow \frac{d^2z}{dt^2} = \frac{d^2y}{dt^2} + L_2\frac{d^2\theta}{dt^2} \qquad \frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} - L_1\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = \frac{1}{L_1}\frac{d^2}{dt^2}(y-x) \Rightarrow \boxed{\frac{d^2z}{dt^2} = -\frac{L_2}{L_1}\frac{d^2x}{dt^2} + \left(1+\frac{L_2}{L_1}\right)\frac{d^2y}{dt^2} \quad (1)}$$

$$F_s = k(x - y - L) \qquad F_d = c\frac{d}{dt}(x - y)$$



$F_y = m a_y$ for mass m_1

$$m_1 \frac{d^2x}{dt^2} = -F_s - F_d - T \qquad (2)$$

$$F_y = m a_y \text{ for } m_2 \Rightarrow m_2 \frac{d^2z}{dt^2} = R_y + T \qquad (3)$$

$$F_x = m a_x \text{ for } m_2 \Rightarrow m_2 \frac{d^2r}{dt^2} = R_x \Rightarrow R_x = 0$$

$\Sigma M_{com} = 0$ for m_2

$$\Rightarrow -T(L_1 + L_2)\overbrace{\cancel{\cos\theta}}^{\sim 1} - R_y L_2 \overbrace{\cancel{\cos\theta}}^{\sim 1} + \overbrace{\cancel{R_x}}^{=0} L_1 \sin\theta = 0$$

$$\Rightarrow R_y = -T(L_1 + L_2)/L_2$$

$$(3) \Rightarrow m_2 \frac{d^2z}{dt^2} = -T\frac{L_1}{L_2} \qquad (4)$$

(1), (2), (4) $\Rightarrow$

$$\boxed{\left(\frac{m_1}{k}+\frac{m_2}{k}\frac{L_2^2}{L_1^2}\right)\frac{d^2x}{dt^2}+\frac{c}{k}\frac{dx}{dt}+x=L+\frac{m_2}{k}\left(\frac{L_2}{L_1}+\frac{L_2^2}{L_1^2}\right)\frac{d^2y}{dt^2}+\frac{c}{k}\frac{dy}{dt}+y}$$

(b) Substitute numbers

$$\frac{1}{10^4}\frac{d^2x}{dt^2}+\frac{1}{10^3}\frac{dx}{dt}+x=L+\frac{0.55}{10^4}\frac{d^2y}{dt^2}+\frac{1}{10^3}\frac{dy}{dt}+y$$

List of standard ODEs for vibration problems

Case I $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+x=C$

Case II $\frac{1}{\alpha^2}\frac{d^2x}{dt^2}-x=-C$

Case III $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C$

Case IV $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+KF(t)$ with $F(t)=F_0\sin\omega t$

Case V $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C+K\left(y+\frac{2\varsigma}{\omega_n}\frac{dy}{dt}\right)$ with $y(t)=Y_0\sin\omega t$

Case VI $\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\varsigma}{\omega_n}\frac{dx}{dt}+x=C-\frac{K}{\omega_n^2}\frac{d^2y}{dt^2}$ with $y(t)=Y_0\sin\omega t$

Case VII $\boxed{\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right) \text{ with } y(t)=Y_0\sin\omega t}$

Compare coefficients:

$$\frac{1}{\omega_n^2}=\frac{1}{10^4} \qquad \frac{2\zeta}{\omega_n}=\frac{1}{10^3} \qquad \frac{\lambda^2}{\omega_n^2}=\frac{55}{10^6}$$

$\Rightarrow \omega_n = 100$ rad/s

$\zeta = 0.05$

$\lambda = 0.742$

$K = 1$

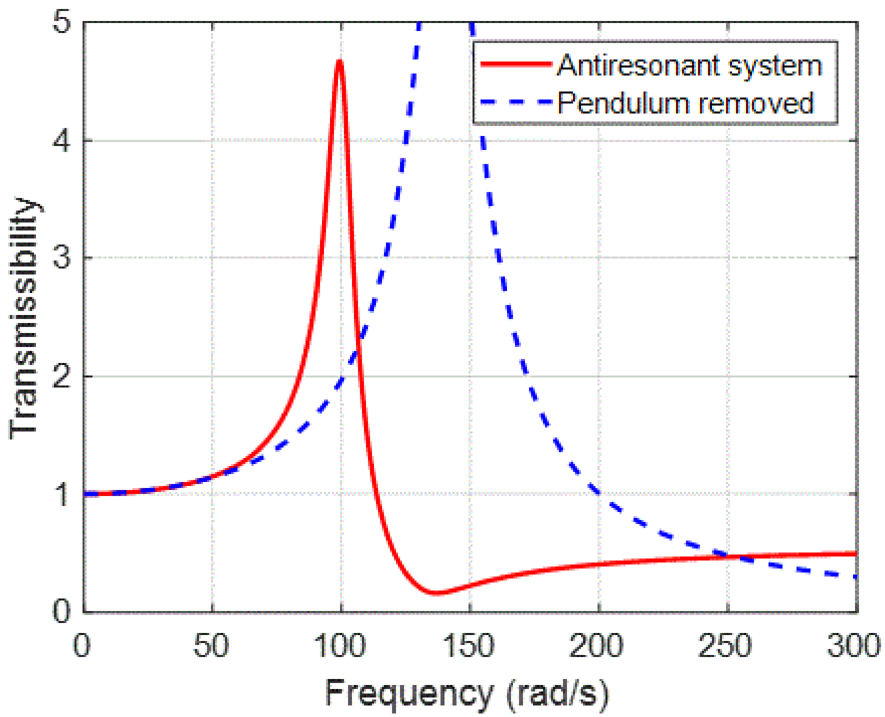
Steady-State Solution to Case VII (From pdf on website)

Equation $$\frac{1}{\omega_n^2}\frac{d^2x}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dx}{dt}+x=C+K\left(\frac{\lambda^2}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y\right)$$ $$y(t)=Y_0\sin(\omega t)$$

$$X_0=KY_0M(\omega/\omega_n,\zeta,\lambda)\qquad M(\omega/\omega_n,\zeta,\lambda)=\frac{\left\{\left(1-\lambda^2\omega^2/\omega_n^2\right)^2+\left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}}$$

$$\phi=\cos^{-1}\frac{1-\lambda^2\omega^2/\omega_n^2}{\left\{\left(1-\lambda^2\omega^2/\omega_n^2\right)^2+\left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}}-\cos^{-1}\frac{1-\omega^2/\omega_n^2}{\left\{\left(1-\omega^2/\omega_n^2\right)^2+\left(2\zeta\omega/\omega_n\right)^2\right\}^{1/2}}\qquad(-\pi<\phi<0)$$

"Transmissibility" = X_0/Y_0 = M



Plot graph w/ MATLAB

Note antiresonance !